Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 5, 1999

THERMOMECHANICS OF ELASTOPLASTIC AND SUPERPLASTIC DEFORMATION OF METALS

A. A. Markin

UDC 539.324;539.376

Using the process theory of A. A. Il'yushin, we consider the problem of determining the thermomechanical parameters of a material element for specified deformation and temperature-variation processes with allowance for the elastic, plastic, and viscous properties of superplastic deformation. The relations obtained are applicable for the case of arbitrary stresses and finite strains. The strain and stress measures are decomposed into elastic, plastic, and viscous components by classifying the processes into reversible, irreversible equilibrium, and nonequilibrium processes.

1. Basic Thermomechanical Relations. A finite deformation process is specified by the law of time variation of the measure $\hat{K}(t)$, whose generalized Jaumann derivative coincides with the strain-rate tensor [1]. The spherical component θ (convolution of $\hat{K}$ with unit tensor $\hat{E}$) is related to the volume change for the element by

$$\frac{dv}{dv_0} = \exp\theta,$$

where dv_0 and dv are the initial and current volumes, respectively.

A change in the deviatoric component $\tilde{K}(t)$ corresponds to a distortion of the element. The coordinates $\tilde{K}$ in a polar basis specify the distortion vector $\boldsymbol{k}$ in Il'yushin's five-dimensional space [2]. The end of the vector $\boldsymbol{k}(t)$ describes the distortion trajectory with the arc coordinate $s(t)$. The distortion rate is expressed in terms of the deviatoric strain rate $\tilde{W}$ by the relation $|\dot{\boldsymbol{k}}| = \dot{s} = |\tilde{W}\cdot\cdot\tilde{W}|^{1/2}$.

The loading process is described by the generalized tensor of actual stresses $\hat{\sigma}(t)$, which is related to $\hat{K}(t)$ by the following expression for the specific (per unit initial volume) elementary work of stresses:

$$d'A = \hat{\sigma}\cdot\cdot\overset{\Delta}{d}\,\hat{K} = \boldsymbol{\tau}\cdot d\boldsymbol{k} + \sigma_0\,d\theta.$$

Here $\boldsymbol{\tau}$ is a loading vector that corresponds to the deviatoric component $\tilde{\sigma}$ and $\sigma_0 = (1/3)\hat{\sigma}\cdot\cdot\hat{E}$ is the hydrostatic stress component. The delta stands for the generalized Jaumann derivative.

The external thermal action on the element over time Δt is characterized by the increment in thermal energy $d'Q$. The differentials of the state parameters are related by the thermodynamic identity [3]

$$d\Psi + S\,dt - d'A + d'w \equiv 0, \tag{1.1}$$

where Ψ is the free energy, S is the specific entropy, and $d'w$ is the dissipation in the elementary volume.

According to the second law of thermodynamics, the entropy differential is expressed in terms of the increments in thermal energy and dissipation as follows:

$$TdS = d'Q + d'w, \qquad d'w \geqslant 0. \tag{1.2}$$

The isotropy postulate implies that, at $t = t^*$, the stress state of an isotropic material element that is unstressed at $t = t_0$ is determined by the laws of deformation $\hat{K}(t)$ and temperature variation $T(t)$ specified

---

Tula State University, Tula 300600. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 5, pp. 164–172, September–October, 1999. Original article submitted January 22, 1998.

in the time interval $[t_0, t^*]$. Generally, the changes in free energy and entropy are also functionals of the deformation process.

The reaction of the material depends on its properties and on the course of the deformation and temperature-variation processes. Hereinafter, we shall distinguish nonequilibrium, irreversible equilibrium, and reversible equilibrium processes.

For an adiabatic process and a constant strain state, the values of all thermomechanical characteristics attained in the equilibrium process should be preserved however long. If the rate of distortion and the rate of change in volume and temperature tend to zero, the process approaches an equilibrium process.

Thus, any nonequilibrium distortion process corresponds to an equilibrium process with the same trajectory in Il'yushin's space. As the time parameter one can use any monotonically increasing parameter, for example, the distance traveled along the distortion trajectory. Below, the subscript "eq" at the deviator and the stress vector denotes an equilibrium process.

An equilibrium process of transition from state T_M, $\hat{\sigma}_M$, $\hat{K}_M$, Ψ_M, and S_M to state T_N, $\hat{\sigma}_N$, $\hat{K}_N$, Ψ_N, and S_N is called a reversible process if, for any reverse transition (with $\hat{\sigma}$ and T varying arbitrarily) from T_N and $\hat{\sigma}_N$ to T_M and $\hat{\sigma}_M$, the values of $\hat{K}$, Ψ, and S that correspond to the process closed for stresses and temperature coincide with the initial values $\hat{K}_M$, Ψ_M, and S_M.

2. Models for Equilibrium and Nonequilibrium Deformation. We introduce certain assumptions on the properties of materials under elastoplastic and superplastic deformation.

1. Materials remain isotropic during deformation.

2. The volume change is reversible (in any process, including ones closed for stresses and temperature, the volume of a material element does not change).

3. In the five-dimensional space of equilibrium stresses $\hat{\sigma}_{\text{eq}}$ there is a closed surface $\Phi(\boldsymbol{\tau}_{\text{eq}}, T) = 0$. If the end of the equilibrium-stress vector $\boldsymbol{\tau}_{\text{eq}}$ is inside or on the surface and $\dot{\boldsymbol{\tau}}_{\text{eq}} \cdot \operatorname{grad} \Phi \leqslant 0$, the process is reversible and the surface remains unchanged. If the end of the vector $\boldsymbol{\tau}_{\text{eq}}$ is on the surface and the condition $\dot{\boldsymbol{\tau}}_{\text{eq}} \cdot \operatorname{grad} \Phi > 0$ is satisfied, the process is irreversible and the surface changes.

4. The initial reversibility surface is a sphere of radius τ_0; the intensity of equilibrium stresses is bounded:

$$\Phi\Big|_{t=t_0} = \tau - \tau_0(T) = 0, \qquad \tau_{\text{eq}} < \tau_s(T).$$

In accordance with the assumptions introduced, the current state $\boldsymbol{\tau}_{\text{eq}}^{(s)}$, $\boldsymbol{k}_s$, and T_s with a surface $\Phi_s(\boldsymbol{\tau}_{\text{eq}}, T_s) = 0$ corresponds to the irreversible (plastic) component $\boldsymbol{k}_{\text{eq}}$ of the distortion vector. The component $\boldsymbol{k}_{\text{eq}}$ is defined as the value of $\boldsymbol{k}$ obtained in the reversible transition at constant temperature from the point $\boldsymbol{\tau}_{\text{eq}}^{(s)}$ to the point of the stress space at which $\boldsymbol{\tau} = \mathbf{0}$. By the definition of reversibility, the value of $\boldsymbol{k}_{\text{eq}}$ is independent of the shape of the trajectory corresponding to this transition and it is a measure of irreversibility for the state considered. For repeated reversible processes, the plastic component remains unchanged:

$$\dot{\boldsymbol{k}}_{\text{eq}} \equiv \mathbf{0}. \tag{2.1}$$

We denote by $\boldsymbol{k}_e$ the varying (elastic) component of the vector $\boldsymbol{k}$ in reversible deformation. At an arbitrary point $\boldsymbol{\tau}$ in the reversibility region, $\boldsymbol{k}_{\text{eq}}$ is given by

$$\boldsymbol{k}_e = \boldsymbol{k} - \boldsymbol{k}_{\text{eq}}. \tag{2.2}$$

It follows from (2.1) that $\boldsymbol{k} = \boldsymbol{k}_e$ in the reversible region. When the process is irreversible, the plastic component changes; therefore, from (2.2) with $\dot{\boldsymbol{\tau}}_{\text{eq}} \cdot \operatorname{grad} \Phi > 0$, we obtain

$$\dot{\boldsymbol{k}} = \dot{\boldsymbol{k}}_e + \dot{\boldsymbol{k}}_{\text{eq}}. \tag{2.3}$$

In connection with the assumption of conservation of the isotropy of elastic properties, we assume that the free energy at a fixed point of the reversible region is independent of the plastic strain component. We write the total energy in the form

$$\Psi = \Psi(\boldsymbol{k}_e, \theta, T). \tag{2.4}$$

Considering an elementary equilibrium process closed for stresses and temperature, we find the increment in dissipation $d'w$ at the irreversible stage:

$$d'w_{\text{eq}} = \boldsymbol{\tau}_{\text{eq}} \cdot d\boldsymbol{k}_{\text{eq}} = \tilde{\sigma}_{\text{eq}} \cdot\cdot d\tilde{K}_{\text{eq}}. \tag{2.5}$$

We define the viscous component of the stress deviator $\tilde{\sigma}_V$ as the difference between the nonequilibrium and equilibrium values of the deviator at the same point of the distortion trajectory

$$\tilde{\sigma}_V = \tilde{\sigma} - \tilde{\sigma}_{\text{eq}}. \tag{2.6}$$

Denoting the difference between the increments in dissipation for nonequilibrium and equilibrium processes by $d'w_V$, we have

$$d'w = d'w_{\text{eq}} + d'w_V. \tag{2.7}$$

Expressions (2.3)–(2.7) are substituted into the thermodynamic identity (1.1) to give

$$\tilde{\sigma}_{\text{eq}} = \frac{\partial \Psi}{\partial \tilde{K}_e}, \qquad \sigma_0 = \frac{\partial \Psi}{\partial \theta}, \qquad S = -\frac{\partial \Psi}{\partial T}, \qquad d'w_V = \tilde{\sigma}_V \cdot\cdot d'\tilde{K}, \tag{2.8}$$

where $\tilde{K}_e$ is the elastic component of the measure $\tilde{K}$.

The second thermodynamic equation (1.2) yields the increment in thermal energy:

$$d'Q = -T\left(\frac{\partial^2 \Psi}{\partial T^2}\,dT + \frac{\partial^2 \Psi}{\partial T\,\partial \tilde{K}_e} \cdot\cdot d\tilde{K}_e + \frac{\partial^2 \Psi}{\partial T\,\partial \theta}\right) - d'w_{\text{eq}} - d'w_V. \tag{2.9}$$

Relations (2.8) and (2.9) must be supplemented by equations that define the plastic component of the distortion vector and the viscous component of the stress vector. We use the maximum principle for the dissipation production [2], according to which the actual direction of the stress vector corresponds to the maximum value of the dissipation rate. We assume that this principle is valid for both the plastic and viscous components of dissipation. It follows from (2.5) that the rate w_{eq} is maximal if the vectors $\boldsymbol{\tau}_{\text{eq}}$ and $\boldsymbol{k}_{\text{eq}}$ have the same direction. Consequently, we have

$$\boldsymbol{\tau}_{\text{eq}}/\tau_{\text{eq}} = \dot{\boldsymbol{k}}_{\text{eq}}/\dot{s}_{\text{eq}}, \tag{2.10}$$

where $\dot{s}_{\text{eq}} = |\dot{\boldsymbol{k}}_{\text{eq}}|$ is the plastic strain rate.

From (2.10) and (2.5), we obtain the following evolution equation for $\boldsymbol{k}_{\text{eq}}$:

$$\boldsymbol{\tau}_{\text{eq}} = \frac{\dot{w}_{\text{eq}}}{\dot{s}^2_{\text{eq}}}\,\dot{\boldsymbol{k}}_{\text{eq}}. \tag{2.11}$$

From the expression for the production of the viscous dissipation component and (2.8), we find the viscous component of the loading vector

$$\boldsymbol{\tau}_V = \frac{\dot{w}_V}{\dot{s}^2}\,\dot{\boldsymbol{k}}. \tag{2.12}$$

The vector properties of the material are defined by Eqs. (2.11) and (2.12), and the scalar properties depend on w_{eq} and w_V, which are functionals of the deformation and temperature-variation processes. Moreover, since the functional of the plastic-dissipation component is invariant with respect to the choice of the time parameter, it can be specified on the plastic deformation trajectory. Hence, we have

$$w_{\text{eq}}(\dot{s}_{\text{eq}}) = A_{\text{eq}}[\dot{\boldsymbol{k}}_{\text{eq}}(s_{\text{eq}}), T(s_{\text{eq}})]_0^{\dot{s}_{\text{eq}}}, \tag{2.13}$$

where A_{eq} is a functional operator that is universal for the given material.

The functional w_V depends heavily on the distortion law $s = s(t)$. Therefore, it can be generally written as

$$w_V(t^*) = A_V[\boldsymbol{k}(t), T(t)]_{t_e}^{t^*}, \tag{2.14}$$

where t_e is the moment of exit into the irreversible region and A_V is a "viscous" operator.

Thus, if the free-energy function (2.4) and the dissipation functionals (2.13) and (2.14) are specified, we obtain a closed thermodynamic model for the material. It comprises the system of evolution equations (2.5)–(2.9), (2.11), and (2.12) and allows one to predict the material response to various thermodynamic actions. In particular, the primal thermodynamic problem is to find the laws of variation of the thermodynamic parameters $\tilde{\sigma}(t)$, $\Psi(t)$, $S(t)$, $w(t)$, and $Q(t)$ for specified deformation and temperature-variation laws.

3. Concrete Definition of the Model of Elastoplastic and Superplastic Deformation. We assume that the rate of plastic dissipation is a function of the plastic trajectory length and the temperature:

$$\dot{w}_{\text{eq}} = \tau_{\text{eq}}(s_{\text{eq}}, T)\dot{s}_{\text{eq}}. \tag{3.1}$$

The presence of the factor $\dot{s}_{\text{eq}}$ is required to satisfy condition (2.13) in integrating (3.1) with respect to time.

We regard the rate of viscous dissipation as a function of s_{eq}, distortion rate, and temperature:

$$\dot{w}_V = \dot{w}_V(s_{\text{eq}}, \dot{s}, T). \tag{3.2}$$

From (2.8) and (2.11) and (3.1), we find the elastic and plastic components of the deviatoric strain rate. Substituting these into (2.4), we arrive at the following equations describing equilibrium deformation:

$$\left(\frac{\partial\Psi}{\partial\tilde{K}_e}\right)^{-1} \cdot \tilde{\sigma}_{\text{eq}}^{\Delta} + \frac{\partial s_{\text{eq}}}{\partial\tau_{\text{eq}}}\,\frac{\dot{\tau}_{\text{eq}}}{\tau_{\text{eq}}}\,\tilde{\sigma}_{\text{eq}} = \tilde{W}. \tag{3.3}$$

According to (2.12) and (3.2), the viscous component of the stress deviator has the form

$$\tilde{\sigma}_V = \tau_V(s_{\text{eq}}, \dot{s}, T)\,\frac{\tilde{W}}{\dot{s}}. \tag{3.4}$$

Integrating Eqs. (3.3) and (3.4) for known $\hat{W}(t)$ and $T(t)$ and specified initial conditions, we obtain time dependences of the deviatoric components of the stress tensor. The time dependence of the spherical component is found from (2.8).

We formulate the initial conditions and conditions of integration in the reversible and irreversible regions. A solution (elastic) in the reversible region follows from (3.3) for $\hat{W}_{\text{eq}} = 0$ and it is valid provided that $\tilde{\sigma}\cdot\cdot\tilde{\sigma} \leqslant \tau_0^2$. An equilibrium solution is obtained by integrating (3.3) subject to the initial condition $\tilde{\sigma}_{\text{eq}}\big|_{t=t_0} = \tilde{\sigma}_0$, where $\tilde{\sigma}_0$ is the deviatoric component on the initial surface of reversibility.

Integration of (3.3) over the segment $[t_0, t]$ yields the law of variation for the equilibrium-stress component $\hat{\sigma}_{\text{eq}}(t)$. It should be noted that the variation of the thermomechanical parameters of the equilibrium process is independent of the nonequilibrium-stress component. However, the viscous-stress component depends on s_{eq} and, hence, on the solution (3.3). Summing up the plastic and viscous components, we obtain the law of variation of the stress deviator in the nonequilibrium process considered.

In studying real processes, it is necessary to have specific representations of laws (2.4), (3.1), and (3.2). We write the free energy as is done in the linear theory of thermoelasticity [3]:

$$\Psi = -S_0T - b(T - T_0)\theta - \frac{1}{2}\,\frac{c(T - T_0)^2}{T_0} + \frac{1}{2}K\theta^2 + Gk_e^2, \tag{3.5}$$

where S_0 is the entropy in the undeformed state, b is a coefficient that takes into account the volume change due to temperature change, c is the heat capacity, K is the bulk modulus, and G is the shear modulus.

Taking into account that the stress intensity must be bounded in equilibrium deformation, we write the plastic hardening law in the form

$$s_{\text{eq}} = \frac{\tau_s - \tau_0}{2G_{\text{eq}}}\ln\left(\frac{\tau_s - \tau_0}{\tau_s - \tau_{\text{eq}}}\right), \tag{3.6}$$

TABLE 1

i	$\dot{s}_i \cdot 10^3$, sec^{-1}	σ_i, MPa	m_i
0	—	—	0.40
1	0.4	14	0.38
2	2.5	28	0.33
3	5.0	35	0.32
4	7.5	40	—

where $\tau_s(T)$ is the limit of plastic hardening (yield point), which depends on temperature, and $2G_{\text{eq}} = (\partial\tau_{\text{eq}}/\partial s_{\text{eq}})\big|_{s_{\text{eq}}=0}$ is the initial tangent modulus of the hardening curve.

To describe the viscous properties of superplastic materials, we use the experimental data of [4–6]. According to these data, high-rate hardening is observed in a bounded range of distortion rates. This condition is satisfied by the following approximation of the high-rate hardening law:

$$\tau_V = \tau_{\text{eq}}(s_{\text{eq}}, T)[\exp(2m_0 \arctan(\dot{s}/\dot{s}_0)) - 1]. \tag{3.7}$$

To clarify the meaning of the parameters $m_0(T)$ and $\dot{s}_0(T)$ in (3.7), we consider a simple loading process in which the direction of the stress vector $\boldsymbol{\tau}$ remains unchanged. In this case, we have $\tau = \tau_V + \tau_{\text{eq}}$. Moreover, at the stage where there is no plastic hardening and $\tau_{\text{eq}} \cong \tau_s$, the scalar properties of the material are described, according to (3.7), by the expression

$$\tau = \tau_s(T)\exp(2m_0 \arctan(\dot{s}/\dot{s}_0)). \tag{3.8}$$

We write relation (3.8) in the logarithmic coordinates $\ln(\tau/\tau_s)$ and $\ln(\dot{s}/\dot{s}_m)$. The high-rate hardening modulus is

$$m = \frac{d\ln(\tau/\tau_s)}{d\ln(\dot{s}/\dot{s}_0)} = \frac{2m_0(\dot{s}/\dot{s}_0)}{1 + (\dot{s}/\dot{s}_0)^2} \tag{3.9}$$

and it takes a maximum value m_0 for $\dot{s} = \dot{s}_0$. Thus, m_0 and $\dot{s}_0$ are the modulus and rate of superplasticity, respectively.

In processing experimental data on extension–compression, it is convenient to write m in the form $m = d\ln(\sigma/\sigma_1)/d\ln(\dot{s}/\dot{s}_1)$, where σ_1 and $\dot{s}_1$ are known values of the stress and the corresponding distortion rate in extension–compression.

Table 1 lists the experimental data of [4] on the extension of samples of AMG6 alloy at $T = 450°$C (since the data are taken from diagrams, they are rather approximate). The last column of Table 1 contains estimates of the high-rate hardening modulus calculated via finite increments of stresses and strain rates.

Substituting the values of m_1 and m_2 (see Table 1) into formula (3.9), we obtain a system of two equations defining the modulus and rate of superplasticity. In our case, $m_0 \approx 0.4$ and $\dot{s}_0 \approx 1.95 \cdot 10^{-3}$ sec^{-1}. These estimates agree well with the following values of [6, p. 20] obtained for extension of AMG6 samples at $T = 400°$C: $m_0 \approx 0.38$ and $\dot{s}_0 \approx 1.5 \cdot 10^{-3}$ sec^{-1}.

4. Deformation Process with Plane Stresses. We consider deformation processes that develop under conditions of plane stresses. Let the Cartesian coordinate axes Ox_1 and Ox_2 lie in the plane of stresses; consequently, $\sigma_{i3} = 0$. We write the velocity field in the form

$$V_1 = 2W_{12}(t)x_2(t) + W_{11}x_1, \qquad V_2 = W_{22}x_2. \tag{4.1}$$

The velocity field (4.1) is close to the one observed in experiments on combined loading of thin-walled cylindrical samples. The velocities of points of cross-sections of the sample are written as

$$V_1 = \frac{R_0}{L_0}\frac{\lambda_1}{\lambda_2}\dot{\varphi}x_2 + (\ln\lambda_1)^{\cdot}x_1, \qquad V_2 = (\ln\lambda_2)^{\cdot}x_2, \tag{4.2}$$

where R_0 and L_0 are the initial radius and length of the sample, respectively, λ_1 and λ_2 are the relative changes in the radius and length of the sample, and φ is the angle of rotation of the section $x_2 = L$.

Differentiating the stress tensor with respect to time using the Jaumann derivative, we have

$$\hat{\sigma}^{\Delta} = \dot{\hat{\sigma}} + \hat{\omega} \cdot \hat{\sigma} - \hat{\sigma} \cdot \hat{\omega}, \tag{4.3}$$

where the nonzero components of the rotation tensor $\hat{\omega}$ are $\omega_{12} = -\omega_{21} = (1/2)(V_{1,2} - V_{2,1})$. Assuming that the material is incompressible, from (3.3), (3.5), (3.6), and (4.1)–(4.3), we obtain the following system of equations describing the process of equilibrium distortion:

$$\begin{gathered}\frac{1}{2G}\dot{\tilde{\sigma}}_{\text{eq}11} + \frac{r\lambda_1}{2\lambda_2}\frac{\sigma_{\text{eq}12}}{2G} + \dot{\lambda}\tilde{\sigma}_{\text{eq}11} = (\ln\lambda_1)^{\cdot}, \qquad \frac{1}{2G}\dot{\tilde{\sigma}}_{\text{eq}22} - \frac{r\lambda_1}{2\lambda_2}\frac{\sigma_{\text{eq}12}}{2G} + \dot{\lambda}\tilde{\sigma}_{\text{eq}22} = (\ln\lambda_2)^{\cdot}, \\ \frac{1}{2G}\dot{\tilde{\sigma}}_{\text{eq}12} + \frac{r\lambda_1}{4\lambda_2}(\tilde{\sigma}_{\text{eq}22} - \tilde{\sigma}_{\text{eq}11}) = \frac{r\lambda_1}{2\lambda_2}\dot{\varphi},\end{gathered} \tag{4.4}$$

where $r = R_0/L_0$ and $\dot{\lambda} = (\tau_s - \tau_0) \cdot \dot{\tau}_{\text{eq}}/(2G_{\text{eq}}(\tau_s - \tau_{\text{eq}}))$.

In particular, if the laws of motion of the end cross section $\varphi(t)$, $\lambda_1(t)$, and $\lambda_2(t)$ and the temperature variation are specified, integration of system (4.4) gives the laws of variation for the equilibrium components of the stress tensor. Then, from (3.4) and (3.7), we determine the viscous components and the change in the stress state.

As an example, we consider extension along the Ox_2 axis with subsequent torsion. In this case, $\sigma_{11} = \sigma_{33} = 0$ and the deviatoric components have the form

$$\tilde{\sigma}_{11} = \tilde{\sigma}_{33} = -\frac{1}{3}\sigma_{22}, \qquad \tilde{\sigma}_{22} = \frac{2}{3}\sigma_{22}. \tag{4.5}$$

Assuming that plastic hardening is almost absent at the extension stage, we have $\tau_{\text{eq}} \to \tau_s$ and $\dot{\tau}_{\text{eq}} \to 0$. The coefficient $\dot{\lambda}$ becomes undetermined and the system degenerates into a flow equation of the Prandtl–Reuss type. According to (4.5), the yield condition has the form

$$\tau_s = \Big(\frac{2}{3}\sigma_{\text{eq}22}^2 + 2\sigma_{\text{eq}11}^2\Big)^{1/2}. \tag{4.6}$$

Condition (4.6) is satisfied identically if one introduces the angular parameter α and writes the stress components as

$$\sigma_{\text{eq}22} = \sqrt{\frac{3}{2}}\,\tau_s\cos\alpha, \qquad \sigma_{\text{eq}12} = \frac{\tau_s}{\sqrt{2}}\sin\alpha. \tag{4.7}$$

Substituting (4.5) and (4.7) into system (4.4) and assuming that the length of the sample remains unchanged in torsion ($\lambda_2 = \text{const}$), we obtain the following system of two differential equations for $\alpha(\varphi)$ and $\lambda_1(\varphi)$:

$$\frac{d}{d\varphi}(\lambda_1^{-1}) = -\frac{r\tau_s}{8\sqrt{2}G\lambda_2}\sin\alpha; \tag{4.8}$$

$$(r\lambda_1)^{-1}\lambda_2\frac{d\alpha}{d\varphi} = -\frac{\sqrt{2}G}{\tau_s}\cos\alpha - \frac{\sqrt{3}}{4}. \tag{4.9}$$

It follows from (4.8) that for real materials, for example, aluminum alloys ($\tau_s/G \approx 10^{-3}$), the parameter λ_1 remains constant during torsion. In integrating (4.9), we assume that the value of λ_1 obtained in tension, which is equal, by virtue of incompressibility, to $\lambda_2^{-1/2}$, is retained. Neglecting terms of order τ_s/G compared to unity, integrating (4.9), and adding the viscous components, we obtain the laws of variation for the stress components at the torsion stage:

$$\sigma_{22} = \sqrt{\frac{3}{2}}\,\tau_s\frac{2\exp(k\varphi)}{\exp(2k\varphi) + 1}, \quad \sigma_{12} = \frac{\tau_s}{\sqrt{2}}\Big(\frac{1 - \exp(2k\varphi)}{1 + \exp(2k\varphi)} + \exp(2m_0\arctan(\dot{s}/\dot{s}_0))\Big),$$

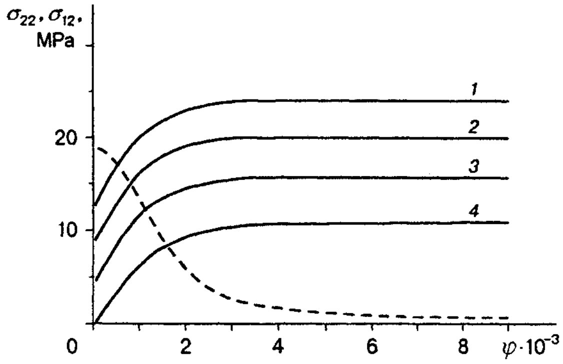

Fig. 1

where $k = \sqrt{2} r \lambda_2^{-3/2} G/\tau_s$ and $\dot{s} = \lambda_2^{-3/2} \dot{\varphi}/\sqrt{2}$.

We note that the same result is obtained if the absolute derivative with respect to time is used instead of the Jaumann derivative. Consequently, the choice of the derivative does not affect stresses at the yield stage.

Figure 1 shows the stresses σ_{22} (dashed curve) and σ_{12} (solid curves) versus the angle of rotation. The diagrams are plotted for AMG6 material with $T = 450°\mathrm{C}$, $\lambda_2 = 1.2$, $r = 0.25$, $m_0 = 0.4$, $\dot{s}_0 = 1.95 \cdot 10^{-3}\ \mathrm{sec}^{-1}$, $\tau_s = 15$ MPa, and $G = 2.7 \cdot 10^4$ MPa (curves 1–3 correspond to the relative strain rates $\dot{s}_1/\dot{s}_0 = 1$, $\dot{s}_2/\dot{s}_0 = 10^{-1}$, and $\dot{s}_3/\dot{s}_0 = 10^{-2}$, respectively; curve 4 shows the variation of σ_{12} in the equilibrium process). One can see that, regardless of the strain rate, the tensile stress decreases rapidly as φ increases, and this supports the well-known lag effect for the vector properties. The fact that σ_{22} is independent of the rate is due to the absence of additional axial strain in torsion. The shear stresses rapidly reach stationary values, which increase with increase in the strain rate.

5. Comparison with the Determining Relation of Smirnov [6]. Smirnov [6] suggested that the deformation process be described by the following expression (in our notation):

$$\tilde{W} = (\tau_m/k_V)^{1/m_V} (\tau - \tau_s)^{1/m_V} (\tau_m - \tau)^{-1/m_V} \frac{\tilde{\sigma}}{\tau}. \tag{5.1}$$

Here k_V and m_V are constant parameters in an isothermal process.

From (5.1), we find the following relation between the stress intensity and the strain rate:

$$\tau = (\dot{s}^{m_V} \tau_m k_V + \tau_s \tau_m)/(\dot{s}^{m_V} k_V + \tau_m). \tag{5.2}$$

In contrast to relations (3.3)–(3.6), formula (5.1) takes into account only rate hardening and does not describe the lag effect of the vector properties and the stages of elastic loading and unloading. Thus, Eqs. (3.3)–(3.6) are more universal, and this is important in describing nonuniform deformation in which the different parts of a body can be in elastic, elastoplastic, and superplastic states at the same time.

It is of interest to compare law (5.2) with expression (3.8). The latter is valid, as (5.2), at the stage where plastic hardening is absent. Both approaches take into account the bounded influence of the strain rate on stresses. Indeed, it follows from (5.2) that $\lim\limits_{\dot{s} \to 0} \tau = \tau_s$ and $\lim\limits_{\dot{s} \to \infty} \tau = \tau_m$. Moreover, from (3.8), we have $\lim\limits_{\dot{s} \to \infty} \tau = \tau_s$ and

$$\lim_{\dot{s} \to \infty} \tau = \tau_m = \tau_s \exp(m_0 \pi). \tag{5.3}$$

In studies dealing with the superplasticity phenomenon, τ_s is called the critical stress and τ_m the yield point.

The four parameters τ_s, τ_m, k_V, and m_V in (5.2) should be determined. The parameters k_V and m_V have no simple mechanical meaning. In particular, m_V does not coincide with the maximum value of the high-rate hardening modulus (3.9).

In (3.8), one should determine experimentally the three constants τ_s, m_0, and s_0, which have an obvious physical meaning. We note that, in our case, the critical stress and the yield point are related by (5.3). On the one hand, this reduces the potentials of approximation, but, on the other hand, instead of τ_s, one can use the more reliable value τ_m. It is well known that experiments on direct determination of the critical stress are very labor consuming and lead to considerable scatter in results. Smirnov [6] gives values $\sigma_m = 3$ MPa for Al-33% Cu alloy at $T = 520$°C and $\sigma_s = 0.3$ MPa at $T = 477$°C and $m_0 = 0.63$. It follows from (5.3) that $\sigma_m/\sigma_s \approx 7.4$. For Mg-6%Zn-6.5% Zr alloy, we have $\sigma_m = 4$, $\sigma_s = 1.4$, and $m_0 = 0.46$ MPa. From (5.3), we find that $\sigma_m/\sigma_s \approx 4.5$. Although the values of the critical stress are not reliable, formula (5.3) gives satisfactory results.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 97-01-00321).

REFERENCES

1. A. A. Markin and L. A. Tolokonnikov, "Measures and determining relations for finite elastoplastic deformation. Methods and Solutions," in: *Applied Problems of Strength and Plasticity* [in Russian], Gor'kii Univ. (1987), pp. 32–37.
2. A. A. Markin, "Change in elastic and plastic properties in finite deformation," *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela*, No. 2, 120–126 (1990).
3. A. A. Il'yushin, *Plasticity. Foundations of General Mathematical Theory* [in Russian], Izd. Akad. Nauk SSSR, Moscow (1963).
4. B. V. Gorev, A. A. Ratnichkin, and O. V. Sosnin, "Deformation mechanism for materials under conditions close to superplasticity," *Probl. Prochnosti*, No. 11, 36–41 (1987).
5. B. V. Gorev, A. A. Ratnichkin, and O. V. Sosnin, "Deformation mechanism for materials under conditions close to superplasticity. 2. Plane stresses," *ibid.*, pp. 42–47.
6. O. M. Smirnov, *Plastic Metal Working in the Region of Superplasticity* [in Russian], Mashinostroenie, Moscow (1979).